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NORMAL SUBGROUPS OF FINITE INDEX FOR THE GROUP REPRESENTATION OF THE CAYLEY TREE

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Abstract. In this paper we give full description of normal subgroups of index four and six for a group representation of the Cayley tree.

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1. Introduction

There are several thousand papers and books devoted to the theory of groups. But still there are unsolved problems, most of which arise in solving of problems of natural sciences as physics, biology etc. In particular, if configuration of physical system is located on a lattice (in our case on the graph of a group) then the configuration can be considered as a function defined on the lattice. Usually, more important configuration (functions) are periodic ones. It is well-known that if the lattice has a group representation then periodicity of a function can be defined by a given subgroup of the representation. More precisely, if a subgroup, say H, is given, then one can define H- periodic function as a function, which has a constant value (depending only on the coset) on each (right or left) coset of H. So the periodicity is related to a special partition of the group (that presents the lattice on which our physical system is located). There are many works devoted to several kind of partitions of groups (lattices) (see e.g. [1], [3], [4], [6]).

Cayley tree. A Cayley tree (Bethe lattice) Γ^k of order $k \geq 1$ is an infinite homogeneous tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L that of edges (arcs). Two vertices x and y are called nearest neighbors if there exists an edge $l \in L$ connecting them. We will use the notation $l = \langle x, y\rangle$. A collection of nearest neighbor pairs $\langle x, x_1\rangle, \langle x_1, x_2\rangle, ...\langle x_{d-1}, y\rangle$ is called a *path* from x to y. The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x to y.

A group representation of the Cayley tree. Let G_k be a free product of $k+1$ cyclic groups of the second order with generators $a_1, a_2, ...a_{k+1}$, respectively.

It is known that there exists a one to one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k.

To give this correspondence we fix an arbitrary element $x_0 \in V$ and let it correspond to the unit element e of the group G_k. Using $a_1, ..., a_{k+1}$ we numerate the nearest-neighbors of element

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e, moving by positive direction. Now we'll give numeration of the nearest-neighbors of each $a_i, i = 1, ..., k+1$ by $a_i a_j, j = 1, ..., k+1$. Since all a_i have the common neighbor e we give to it $a_i a_i = a_i^2 = e$. Other neighbor are numerated starting from $a_i a_i$ by the positive direction. We numerate the set of all the nearest-neighbors of each $a_i a_j$ by words $a_i a_j a_q, q = 1, ..., k+1$, starting from $a_i a_j a_j = a_i$ by the positive direction. Iterating this argument one gets a one-to-one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k.

In Chapter 1 of [4] it was constructed several normal subgroups of the group representation of the Cayley tree. In particular some normal subgroups of index 4 and index 6 were constructed. In this paper we shall give full description of normal subgroups of index four and index six.

Normal subgroups of finite index of the group G_k. Any (minimal represented) element $x \in G_k$ has the following form: $x = a_{i_1} a_{i_2} ... a_{i_n}$, where $1 \le i_m \le k+1, m = 1, ..., n$. The number n is called the length of the word x and is denoted by $l(x)$. The number of letters $a_i, i = 1, ..., k+1$, that enter the non-contractible representation of the word x is denoted by $w_x(a_i)$.

The following is well-known (see [2],[5]).

Proposition 1.1. *Let φ be homomorphism of the group G_k with the kernel H. Then H is a normal subgroup of the group G_k and $\varphi(G_k) \simeq G_k/H$, (where G_k/H is a quotient group) i.e., the index $|G_k : H|$ coincides with the order $|\varphi(G_k)|$ of the group $\varphi(G_k)$.*

Let H be a normal subgroup of a group G. Define the homomorphism g from G onto the quotient group G/H by $g(a) = aH$ for all $a \in G$. Then $Ker\varphi = H$.

Definition 1.1. *Let $M_1, M_2, ..., M_m$ be some sets and $M_i \ne M_j$, for $i \ne j$. We call the intersection $\cap_{i=1}^m M_i$ contractible if there exists $i_0 (1 \le i_0 \le m)$ such that*

$$\cap_{i=1}^m M_i = \left(\cap_{i=1}^{i_0-1} M_i\right) \cap \left(\cap_{i=i_0+1}^m M_i\right).$$

Let $N_k = \{1, ..., k+1\}$. The following Proposition describes several normal subgroups of G_k. Put

$$H_A = \left\{ x \in G_k \mid \sum_{i \in A} \omega_x(a_i) \text{ is even} \right\}. \tag{1}$$

Proposition 1.2. *[4] For any $\emptyset \ne A \subseteq N_k$, the set $H_A \subset G_k$ satisfies the following properties:*

(a) H_A is a normal subgroup and $|G_k : H_A| = 2$;

(b) $H_A \ne H_B$, for all $A \ne B \subseteq N_k$;

(c) Let $A_1, A_2, ..., A_m \subseteq N_k$. If $\cap_{i=1}^m H_{A_i}$ is non-contractible, then it is a normal subgroup of index 2^m.

2. New normal subgroups of finite index

2.1. **The case of index four.**

Proposition 2.1. *Let φ be an epimorphism of the group G_k onto group $(Z_n, +_n)$, $n \in N$. Then $n \in \{1, 2\}$.*

Proof. Let $a_i \in G_k$, $i \in N_k$. Then $[0]_n = \varphi(e) = \varphi(a_i^2) = (\varphi(a_i))^2$. Hence for the order of $\varphi(a_i)$ we have $o(\varphi(a_i)) \in \{1, 2\}$.

`Case 1.` Let n be an odd number. By Lagrange's theorem, the group $(Z_n, +_n)$ hasn't any element $\varphi(a_j) \in G_k, j \in N_k$ such that $o(\varphi(a_j)) = 2$, i.e., $\varphi(x) = [0]_n$ for any $x \in G_k$. Since $\varphi(G_k) \simeq Z_n$ we have $n = 1$.

`Case 2.` Let n be an even number. Denote

$$A = \{i| \ \varphi(a_i) = [0]_n\}, \quad B = \{i| \ \varphi(a_i) = [n/2]_n\}.$$

Since $a_i^2 = e$ and φ is an epimorphism we note that there is only one element $[c]_n \in Z_n$ such that $o([c]_n) = 2$, i.e., $[c]_n = [n/2]_n = 2$. It is clear that if $c \in G_k$ such that $o(c) = m$ then $o(\varphi(c))$ divides m. Hence $A \cup B = N_k$. Then since $\varphi \simeq Z_n$ we have $n = 2$. Consequently

$$\varphi(x) = \begin{cases} [0]_2, & \text{if } \sum_{i \in B} \omega_x(a_i) \quad \text{is even,} \\ [1]_2, & \text{if } \sum_{i \in B} \omega_x(a_i) \quad \text{is odd.} \end{cases}$$

□

Lemma 2.1. *[2]. Every finite cyclic group of order n is isomorphic to $(Z_n, +_n)$.*

By Proposition 2.1, Lemma 2.1 we get

Corollary 2.1. *Let $\varphi : G_k \to \ < a >$ (finite cyclic group) be an epimorphism of groups. Then $|G_k : Ker\varphi| \in \{1, 2\}$.*

Corollary 2.2. *Any normal subgroup of index 2 has the form (1), i.e, $\{H_A| \ \ A \subseteq N_k\} = \{H| \ \ |G_k : H| = 2\}$.*

Consider the set $G = \{e, a, b, c\}$. Let e be an identity element of G and define $*$ on G by means of the following operation:

$$b * a = a * b = c, \ c * a = a * c = b, \ c * b = b * c = a, \ a^2 = e, \ b^2 = e, \ c^2 = e.$$

This group is well-known as Klein 4- group.

Proposition 2.2. *[2]. There are only two groups of order 4 (up to isomorphism), a cyclic group of order 4 and K_4 (Klein 4-group).*

Theorem 2.1. *Any normal subgroup of index 4 has the form $H_A \cap H_B$, i.e.*

$$\{H| \ \ |G_k : H| = 4\} = \{H_A \cap H_B| \ \ A, B \subseteq N_k, A \neq B\}.$$

Proof. By Proposition 2.1 there is not epimorphism φ of G_k onto $(Z_4, +_4)$. By Proposition 2.2 we have $\varphi(G_k) = K_4$.

(a) By Proposition 1.2 we get

$$\{H_A \cap H_B| \ \ A, B \subseteq N_k, \ \ A \neq B \neq \emptyset\} \subseteq \{Ker\varphi| \ \ |G_k : Ker\varphi| = 4\}.$$

(b) Let $S = \{A_0, A_1, A_2, A_3\}, \ A_i \subset N_k, \ \cup_{i=0}^{3} A_i = N_k$. Here and further on $|S|$ denotes the cardinality of S. It's easy to check $|S| \geq 3$.

`Case 1`. Let $|S| = 3$. If $A_0 \neq \emptyset$ then $|G_k : Ker\varphi| = 2$. Hence there exist $j \in \{1, 2, 3\}$ such that $A_0 = A_j = \emptyset$. Let $j = 3$ (the case $j \in \{1, 2\}$ is similar). Then there exists a unique epimorphism (corresponding to A_1, A_2), i.e.,

$$\varphi_{A_1 A_2}(x) = \begin{cases} e, \text{ if } \sum_{i \in A_1} \omega_x(a_i), \quad \sum_{i \in A_2} \omega_x(a_i) \quad \text{are even,} \\ a, \text{ if } \sum_{i \in A_1} \omega_x(a_i) \text{ is odd,} \quad \sum_{i \in A_2} \omega_x(a_i) \text{ is even,} \\ b, \text{ if } \sum_{i \in A_1} \omega_x(a_i) \text{ is even,} \quad \sum_{i \in A_2} \omega_x(a_i) \text{ is odd,} \\ c, \text{ if } \sum_{i \in A_1} \omega_x(a_i), \quad \sum_{i \in A_2} \omega_x(a_i) \text{ are odd.} \end{cases}$$

Then

$$Ker\varphi_{A_1 A_2} = H_{A_1} \cap H_{A_2} \subset \{H_A \cap H_B| \ \ A, B \subseteq N_k\}.$$

`Case 2`. $|S| = 4$.

`Case 2.1.` Let $A_3=\emptyset$ (the case A_1, A_2 are similar). Then $|A_i|\geq 1,\ \ i\in\{0,1,2\}$ and $\sum_{i=0}^{2}|A_i|=k+1$. For all $x\in G_k$ we can construct following epimorphism

$$\varphi_{A_0A_1A_2}(x)=\begin{cases} e, \text{ if } \sum_{i\in A_1}\omega_x(a_i),\ \sum_{i\in A_2}\omega_x(a_i) \quad \text{are even,} \\ a, \text{ if } \sum_{i\in A_1}\omega_x(a_i) \text{ is odd, } \quad \sum_{i\in A_2}\omega_x(a_i) \text{ is even,} \\ b, \text{ if } \sum_{i\in A_1}\omega_x(a_i) \text{ are even, } \quad \sum_{i\in A_2}\omega_x(a_i) \text{ are odd,} \\ c, \text{ if } \sum_{i\in A_1}\omega_x(a_i),\ \sum_{i\in A_2}\omega_x(a_i) \text{ are odd.} \end{cases}$$

Then

$$Ker\varphi_{A_0A_1A_2}=H_{A_1}\cap H_{A_2}\subset\{H_A\cap H_B|\ A,B\subseteq N_k\}.$$

`Case 2.2.` $|A_0|\geq 0,\ |A_i|\geq 1,\ \ i\in\{1,2,3\}$ and $\sum_{i=0}^{3}|A_i|=k+1$. Here, as before we can construct a unique epimorphism $\varphi_{A_0A_1A_2A_3}$.

$$Ker\varphi_{A_0A_1A_2A_3}=\left\{x|\sum_{i\in A_1\cup A_3}\omega_x(a_i),\ \sum_{i\in A_2\cup A_3}\omega_x(a_i) \text{ are even}\right\}.$$

Hence

$$Ker\varphi_{A_0A_1A_2A_3}=H_{A_1\cup A_3}\cap H_{A_2\cup A_3}\subset\{H_A\cap H_B|\ A,B\subseteq N_k\}.$$

Thus we have proved

$$\{Ker\varphi|\ |G_k:Ker\varphi|=4\}\subseteq\{H_A\cap H_B|\ A,B\subseteq N_k,\ \ A,B\neq\emptyset\}.$$

This completes the proof. □

2.2. **The case of index six.**

Proposition 2.3. *[2]. There are only two (up to isomorphism) groups of order* 6*, a cyclic group of order* 6 *and* S_3.

Let φ is a homomorphism of the group G_k. Then by Proposition 2.3 we get following Corollary.

Corollary 2.3. *If* $|G_k:Ker\varphi|=6$ *then* $\varphi(G_k)\simeq(S_3,\circ)$.

Let $\Xi_n=\{A_1,A_2,...,A_n\}$ be a partition of the set N_k/A_0, where $A_0\subset N_k,\ \ 0\leq|A_0|<k+1$. Put m_j is a minimal element of $A_j,\ \ j\in\{1,2,...,n\}$.

Now we'll define an equivalence relation on the set G_k. Let $x=a_{i_1}a_{i_2}...a_{i_q}\in G_k$. If $i_p\in A_0,\ \ p\in\{1,2,...,q\}$ then we'll put e instead of a_{i_p} and if $i_p\in A_j,\ \ j\in\{1,...,n\}$ then we'll put m_j instead of i_p, i.e.,

$$x=a_{i_1}a_{i_2}...a_{i_p}\to a_{j_1}a_{j_2}...a_{j_p}=a_{m_{l_1}}a_{m_{l_2}}....a_{m_{l_s}}=\tilde{x},\quad s\leq n. \tag{2}$$

$\tilde{x}$ is a non-contractible representation of the word $a_{m_{l_1}}a_{m_{l_2}}....a_{m_{l_s}}$. Introduce the following equivalence relation on the set $G_k : x\sim y$ if $\tilde{x}=\tilde{y}$. It's easy to see this relation is reflexive, symmetric and transitive.

Let $\Xi_n=\{A_1,\ A_2,...,A_n\}$ be a partition of $N_k\backslash A_0,\ \ 0\leq|A_0|\leq k+1-n$. Then we consider function $u_n:\{a_1,a_2,...,a_{k+1}\}\to\{e,a_1,...,a_{k+1}\}$ as

$$u_n(x)=\begin{cases} e, \quad \text{if } \ x=a_i, i\in A_0 \\ a_{m_j}, \ \text{if } \ x=a_i, i\in A_j, j=1,2,...,n. \end{cases}$$

Define $\gamma_n:G_k\to G_k$ by the formula

$$\gamma_n(x)=\gamma_n(a_{i_1}a_{i_2}...a_{i_s})=u_n(a_{i_1})u_n(a_{i_2})...u_n(a_{i_s})$$

Put

$$H_{\Xi_n} = \{x|\ l(\gamma_n(x)) : 6\},\ n < k+1. \tag{3}$$

Proposition 2.4. *Let $\Xi_n = \{A_1,\ A_2, ..., A_n\}$ be a partition of $N_k \backslash A_0,\ 0 \leq |A_0| \leq k+1-n$. Then the following properties hold*

(a) If $\Xi_2 = \{A_1, A_2\}$ then H_{Ξ_2} is a normal subgroup of index six of G_k.

(b) Let $\Xi_3 = \{A_1, A_2, A_3\}$ and $m_1, m_2, m_2m_1m_2$ are minimal elements of A_1, A_2, A_3 respectively then H_{Ξ_3} is a normal subgroup of index six of G_k.

Proof. We'll prove Property (a) (Property (b) is similar). Let $x = a_{i_1}a_{i_2}...a_{i_n} \in G_k$. It's sufficient to show $x^{-1}H_{\Xi_2}x \subseteq H_{\Xi_2}$. Let $l(\tilde{x})$ be odd (the case even is similar) and $h \in H_{\Xi_2}$ then $\tilde{x} = a_{m_{i_1}}a_{m_{i_2}}...a_{m_{i_2}}$ and $\tilde{h} = a_{m_{j_1}}a_{m_{j_2}}...a_{m_{j_2}}$, where $i_1, i_2, j_1, j_2 \in \{1, 2\}$. Let y be the non-contractible representation of the word $\tilde{x}^{-1}\tilde{h}\tilde{x}$. Then $l(y) = l(\tilde{h})$. Since $\widetilde{x^{-1}hx} = y$ we have $\gamma_2(x^{-1}hx) : 6$. Hence $x^{-1}hx \in H_{\Xi_2}$. This completes the proof. □

Let $\varphi : G_k \to S_3$ be an epimorphism. Denote

$$B_0 = \{i|\ \varphi(a_i) = e\},\quad B_1 = \{i|\ \varphi(a_i) = (12)\},$$

$$B_2 = \{i|\ \varphi(a_i) = (13)\},\quad B_3 = \{i|\ \varphi(a_i) = (23)\}. \tag{4}$$

Remark 2.1. *$\{B_1, B_2, B_3\}$ is a partition of $N_k \setminus B_0$.*

Lemma 2.2. *Let $\varphi : G_k \to S_3$ be an epimorphism. For any $x \in G_k$ there exist $y \in G_k$ such that $\varphi(x) = \varphi(y)$ and $l(y) \leq 6$.*

Proof. Let $x = a_{i_1}a_{i_2}...a_{i_n} \in G_k$ and $y = a_{i_1}...a_{i_{n-6s}}$, where $s = \left[\frac{n}{6}\right]$, $a_{i_0} = e$. Then $l(y) \leq 6$. From $(23) = (13)\circ(12)\circ(13)$ and $(12)\circ(13)\circ(12)\circ(13)\circ(12)\circ(13) = e$ we get $\varphi(x) = \varphi(y)$. □

Theorem 2.2. *Let H be a normal subgroup of the group G_k. Then $\{H|\ |G_k : H| = 6\} = \{H_{\Xi_2},\ H_{\Xi_3}\}$.*

Proof. Let $\varphi : G_k \to S_3$ be an epimorphism with $|G_k : Ker\varphi| = 6$. By Corollary 2.3 we get $\varphi(G_k) \simeq S_3$. By Proposition 2.4, $H_{\Xi_2} \subseteq \{H|\ |G_k : H| = 6\}$. Hence it's sufficient to check $\{H|\ |G_k : H| = 6\} \subset H_{\Xi_2}$.

`Case 1`. Let $\Xi_2 = \{A_1, A_2\}$- be a partition of $N_k \setminus A_0$. Then we can construct six epimorphism (corresponding to A_0, A_1, A_2). By Lemma 2.2 we can show these epimorphisms. Put m_i is a minimal element of $B_i,\ i \in \{1, 2, 3\}$, defined in (4).

`Case 1.1.` Let $A_1 = B_1$ and $A_2 = B_2$ ($A_1 = B_2,\ A_2 = B_1$ are similar). Then

$$\varphi^{(1)}_{A_0A_1A_2}(x) = \begin{cases} e,\ \text{ if }\ \tilde{x} \in \{e,\ a_{m_1}a_{m_2}a_{m_1}a_{m_2}a_{m_1}a_{m_2},\ a_{m_2}a_{m_1}a_{m_2}a_{m_1}a_{m_2}a_{m_1}\}, \\ (12),\ \text{ if }\ \tilde{x} \in \{a_{m_1},\ a_{m_2}a_{m_1}a_{m_2}a_{m_1}a_{m_2}\}, \\ (13),\ \text{ if }\ \tilde{x} \in \{a_{m_2},\ a_{m_1}a_{m_2}a_{m_1}a_{m_2}a_{m_1}\}, \\ (23),\ \text{ if }\ \tilde{x} \in \{a_{m_1}a_{m_2}a_{m_1},\ a_{m_2}a_{m_1}a_{m_2}\}, \\ (312),\ \text{ if }\ \tilde{x} \in \{a_{m_1}a_{m_2},\ a_{m_2}a_{m_1}a_{m_2}a_{m_1}\}, \\ (231),\ \text{ if }\ \tilde{x} \in \{a_{m_2}a_{m_1},\ a_{m_1}a_{m_2}a_{m_1}a_{m_2}\}. \end{cases}$$

`Case 1.2.` $A_1 = B_1,\ A_2 = B_3$ ($B_1 = A_2,\ B_3 = A_1$ are similar).

$$\varphi^{(2)}_{A_0A_1A_2}(x)=\begin{cases} e, \ \text{if} \ \tilde{x}\in\{e, \ a_{m_1}a_{m_3}a_{m_1}a_{m_3}a_{m_1}a_{m_3}, \ a_{m_3}a_{m_1}a_{m_3}a_{m_1}a_{m_3}a_{m_1}\}, \\ (12), \ \text{if} \ \tilde{x}\in\{a_{m_1}, \ a_{m_3}a_{m_1}a_{m_3}a_{m_1}a_{m_3}\}, \\ (13), \ \text{if} \ \tilde{x}\in\{a_{m_1}a_{m_3}a_{m_1}, \ a_{m_3}a_{m_1}a_{m_3}\}, \\ (23), \ \text{if} \ \tilde{x}\in\{a_{m_3}, \ a_{m_1}a_{m_3}a_{m_1}a_{m_3}a_{m_1}\}, \\ (312), \ \text{if} \ \tilde{x}\in\{a_{m_3}a_{m_1}, \ a_{m_1}a_{m_3}a_{m_1}a_{m_3}\}, \\ (231), \ \text{if} \ \tilde{x}\in\{a_{m_1}a_{m_3}, \ a_{m_3}a_{m_1}a_{m_3}a_{m_1}\}. \end{cases}$$

`Case 1.3.` $A_1=B_2, \ A_2=B_3$ ($A_1=B_3, \ A_2=B_2$ are similar).

$$\varphi^{(3)}_{A_0A_1A_2}(x)=\begin{cases} e, \ \text{if} \ \tilde{x}\in\{e, \ a_{m_2}a_{m_3}a_{m_2}a_{m_3}a_{m_2}a_{m_3}, \ a_{m_3}a_{m_2}a_{m_3}a_{m_2}a_{m_3}a_{m_2}\}, \\ (12), \ \text{if} \ \tilde{x}\in\{a_{m_2}a_{m_3}a_{m_2}, \ a_{m_3}a_{m_2}a_{m_3}\}, \\ (13), \ \text{if} \ \tilde{x}\in\{a_{m_2}, \ a_{m_3}a_{m_2}a_{m_3}a_{m_2}a_{m_3}\}, \\ (23), \ \text{if} \ \tilde{x}\in\{a_{m_3}, \ a_{m_2}a_{m_3}a_{m_2}a_{m_3}a_{m_2}\}, \\ (312), \ \text{if} \ \tilde{x}\in\{a_{m_2}a_{m_3}, \ a_{m_3}a_{m_2}a_{m_3}a_{m_2}\}, \\ (231), \ \text{if} \ \tilde{x}\in\{a_{m_3}a_{m_2}, \ a_{m_2}a_{m_3}a_{m_2}a_{m_3}\}. \end{cases}$$

Hence

$$Ker\varphi^{(i)}_{A_0A_1A_2}=H_{\Xi_2}.$$

`Case 2.` Let $\Psi_3=\{A_1,A_2,A_3\}$ be a partition of $N_k\setminus A_0$. Then there exist six epimorphism (corresponding to A_1,A_2,A_3). Let $A_i=B_i, \ i\in\{0,1,2,3\}$ (other cases are similar). It's easy to see $Ker\varphi_{A_0A_1A_2A_3}$ is equal to

$$(x\in G_k \mid \tilde{x}\in\{e,a_{m_1}a_{m_2}a_{m_1}a_{m_3},a_{m_1}a_{m_3}a_{m_1}a_{m_2},a_{m_1}a_{m_2}a_{m_3}a_{m_2},a_{m_1}a_{m_3}a_{m_2}a_{m_3},a_{m_2}a_{m_3}a_{m_1}a_{m_3}\}).$$

From $\varphi(a_{m_1}a_{m_2a_{m_1}})=\varphi(a_{m_3})$ we get $Ker\varphi_{A_0A_1A_2A_1}=H_{\Xi_3}$. $\square$

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